647510 **MSC3MatC303x** **Seat No :________**

M.Sc.(Maths.) Semester - 3 (*CBCS*) Examination
Oct/Nov-2021 (NEW COURSE)
Number Theory -1 (Core (New))

Time: 2:30 Hours **Marks: 70**

Instructions:
1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1(a) Find integers x, y, z satisfying $\gcd(198, 288, 512) = 198x + 288y + 512z$. **4 Marks**

Q.1(b) **Answer any two question out of three.** **10 Marks**

(1) Prove that if $d|n$ then $2^d - 1|2^n - 1$.

(2) If n is an odd integer then show that $n^4 + 4n^2 + 11$ is of the form $16k$.

(3) If p_n is the n^{th} prime number then show that $p_n \leq 2^{2^{n-1}}$.

Q.2(a) If $ca \equiv cb \ (mod\ n)$ then show that $a \equiv b \ \left(mod \frac{n}{d}\right)$, where $d = \gcd(c, n)$. **4 Marks**

Q.2(b) **Answer any two question out of three.** **10 Marks**

(1) Let $N = a_m 10^m + a_{m-1} 10^{m-1} + \cdots + a_1 10 + a_0$ be the decimal expansion of the positive integer $N, 0 \leq a_k \leq 10$, and let $T = a_0 - a_1 + a_2 - \cdots + (-1)^m a_m$. Then $11|N$ if and only if $11|T$.

(2) A band of 17 pirates stole a sack of gold coins. When they tried to divide the fortune into equal portions, 3 coins remained. In the ensuing brawl over who should get the extra coins, one pirate was killed. The wealth was redistributed, but this time an equal division left 10 coins. Again an argument developed in which another pirate was killed. But now the total fortune was evenly distributed among the survivors. What was the least number of coins that could have been stolen?

(3) What is the remainder when the sum $1^5 + 2^5 + 3^5 + \cdots + 99^5 + 100^5$ is divided by 4?

Q.3(a) If a has order k modulo n and $h > 0$ then show that a^h has order $\frac{k}{\gcd(h,k)}$ modulo n. **4 Marks**

Q.3(b) **Answer any two question out of three.** **10 Marks**

(1) Show that the odd prime divisors of the integer $n^4 + 1$ are of the form $8k + 1$.

(2) If p is a prime number and $d|p - 1$ then show that the congruence $x^d - 1 \equiv 0 \ (mod\ p)$ has exactly d solutions.

(3) Show that if n has a primitive root then it has exactly $\phi(\phi(n))$ of them.

Q.4(a) If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$ then show that $1 > \frac{n}{\sigma(n)} > \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right)$. **4 Marks**

Q.4(b) **Answer any two question out of three.** **10 Marks**

(1) If the integer $n > 1$ has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ then show that $\sum_{d|n} \mu(d)\tau(d) = (-1)^r$.

(2) Verify that $1000!$ Terminates in 249 zeros.

(3) For $n > 1$ show that the sum of the positive integers less than n and relatively prime to n is $\frac{1}{2} n\phi(n)$.

Q.5(a) State and prove Chinese Remainder Theorem. **4 Marks**

Q.5(b) **Answer any two question out of three.** **10 Marks**

(1) For positive integers a and b show that $\gcd(a, b)\ lcm(a, b) = ab$.

(2) Show that the functions τ and σ both are multiplicative functions.

(3) Show that there is an infinite number of primes.

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